G/G/m/N Queue using Refined Diffusion Approximation Techniques

Rashmita Sharma
Assistant professor
Department of Mathematics,
D.A.V.(P.G.) College, Dehradun (U.K.) India

Abstract
We develop refined diffusion approximation for G/G/m queueing model with finite capacity. Diffusion equation has been obtained by approximating the discrete flow of customers in the system by continuous one. A refined diffusion approximation technique has been developed for the model by imposing reflecting boundaries at 0 & N. approximate formulae for the mean number of customers in the system, delay probability and mean queue length have been derived in this paper.

Keywords: Multi server- queues, approximation , diffusion models

1. Introduction
The study of multi-server queueing models has become very useful with the rapid growth of telephone call centers and more general customer contact centers. Multi-server finite queueing systems are of interest from the view point of many practical applications in flexible manufacturing systems (FMS), but very few results are known for general distribution for arrival and service process. Several authors have been developed various approximation techniques e.g.,see Cosmtatos [2], Stoyan [14], Takahashi [29], Nozaki and Ross [12], Hokstad [5], Sunaga et al. [15], Boxma et al. [1], Yao and Buzacott [28], & Xiong and Altiok [17]. Multi-server queues using diffusion approximation techniques were studied by Halachmi and Franta [3] & [4], Chen and Shanthikumar [26], Whitt [21],[22],[23], [24], & [25], Choi and Shin[13], Kimura [6], [7], [8], [9], [10],& [11], and Yang et al. [31].

Recently diffusion approximation techniques have been used by Dai and He [17], Maheshwari et al. [27], Miyazawa [18] and Weersinghe [19].


Yao and Buzacott [28] analysed G/G/c/N model for a flexible manufacturing system by imposing elementary return boundaries at 0 & N. Choi and Shin [13] also used elementary return process with elementary return boundaries at x=0 and x=N to obtain an approximation of the transient queue.

In this paper, we consider the G/G/m/N model and suggest an approach based on diffusion process similar to Yao [3] refined approach for M/G/m model. We impose two reflecting barriers at 0 & N and provide solution approach to diffusion equations retaining the piecewise continuity of the infinitesimal mean and variance as discussed in Kimura [6].
2. The Model and refined diffusion equations

The G/G/m/N system we deal with is specified under the steady-state conditions by the following assumptions: customers arrive according to a general process at rate $\lambda > 0$ and $C_\sigma$ is the coefficient of variation of arrivals times. Suppose customers are served by one of $m (\geq 1)$ parallel servers in the order of their arrival. The service times of customers are identically and independently distributed (i.i.d.) random variables with mean $\mu^{-1}$ and coefficient of variation $C_\sigma$.

Let $Q(t)$ be the number of customers at time $t$ and $P_n(t)$ be the probability that $Q(t)$ is $n$. Again, suppose that $\pi_k (k=0,1,2,3,\ldots,N)$ be the steady state probabilities of the queueing process $Q(t)$. Consider $\{ X(t)/X(t) \in [0,N]\}$ denotes the diffusion process and $p(x)$ denotes the steady-state probability density function of the process $X(t)$. Let $b(x)$ and $a(x)$ be the infinitesimal mean and variance of the process.

Then we propose

For $k-1 < x \leq k$ \hspace{1cm} (k= 1 \ldots m)

\[
b(x) = \lambda - (\lfloor x \rfloor ^m) \mu = b_k \quad (1)\]

\[
a(x) = \lambda C_\sigma^2 + (\lfloor x \rfloor ^m) \mu C_\sigma^2 = a_k \quad (2)\]

and for $m < x < N$

\[
b(x) = b_m \quad (3)\]

\[
a(x) = a_m \quad (4)\]

Let $\pi_0$ be the probability mass concentrated at $x = 0$. It follows from Yao [31] that $p(x)$ satisfies the following equations:

For $0 < x \leq 1$

\[
P_1''(x) - \gamma_1 P_1(x) = 0 \quad (5)\]

\[
P_1(\frac{1}{2}) = \pi_0 C_1 \quad (6)\]

\[
\lim_{x \to 0} [P_1'(x) - \gamma_1 P_1] = 0 \quad (7)\]

Where $\gamma_1 = 2 \frac{b_1}{a_1}$

Here we treat 0 as a reflecting barrier.

For $k-1 < x \leq k$ \hspace{1cm} (k= 2 \ldots m)

\[
p_k''(x) - \gamma_k p_k'(x) = 0 \quad (8)\]

\[
p_k(k - \frac{1}{2}) = \pi_0 C_k \quad (9)\]
\[ p_k (k = 1 + 0 ) = p_{k-1}^{(k-1)} \quad \text{if } \rho > (1 - \gamma_m)^{-1} \]  
\[ p_k (k) = p_{k+1}^{(k+1)} \quad \text{if } \rho \leq (1 - \gamma_m)^{-1} \]

where \( \gamma_k = \frac{2b_k}{a_k} \) ; \( (k = 2,3,\ldots) \)

\[ \rho = \frac{\lambda}{m} \]

Here \( p(x) \) is continuous everywhere except at either \( x = 1 \) or \( x = m \).

There are discontinuities at \( x = m \) and at \( x = 1 \) for \( \rho > (1 - \gamma_m)^{-1} \) and \( \rho \leq (1 - \gamma_m)^{-1} \) respectively.

For \( m < x < N \)

\[ P_{m+1}'' (x) - \gamma_m P_{m+1}' (x) = 0 \]

\[ P_{m+1}' (m) = \pi_0 c_m \gamma_m \left( \frac{\rho}{(1-\rho)} \right) \]

\[ \lim_{{x \to N}} P_{m+1} (x) = 0 \]

Where \( c_k = \frac{(mp)^k}{k!} \)

And \( \gamma_m < 0 \) when \( \rho > 1 \)

3. The Solution

For \( 0 < x \leq 1 \), we have

\[ P_1 (x) = \begin{cases} 
\pi_0 c_1 e^{\gamma_1(x-\frac{1}{2})} & \text{if } b_1 \neq 0 \\
\pi_0 c_1 & \text{if } b_1 = 0 
\end{cases} \]

In case of \( k-1 < x \leq k \)

\[ p_k (x) = \begin{cases} 
\pi_0 \left[ c_k + \frac{(c_k - q_{k-1})e^{\gamma_k(x-k+\frac{1}{2})} - 1}{1-e^{\gamma_k}} \right] & \text{if } b_k \neq 0 \\
\pi_0 [2(x-k)(q_k - C_k) - q_k] & \text{if } b_k = 0 
\end{cases} \]

(\( k = 2,3,\ldots \))

Again, for \( m < x \leq N \)

\[ P_{m+1} (x) = \frac{\pi_0 \gamma_m c_m \rho}{(1-\rho)(e^{\gamma_m (N-m)} - 1)} \left( 1 - e^{\gamma_m (X-N)} \right) \]

Where \( \gamma_k = \frac{2b_k}{a_k} \)

\[ q_k = \begin{cases} 
c_k + (c_k - q_{k-1})e^{\gamma_k/2} & \text{if } b_k \neq 0 \\
2c_k + q_{k-1} & \text{if } b_k = 0 
\end{cases} \]
Now the steady state probabilities can be approximated as follows:

\[
\pi_1 = \int_{k-1}^{k} p(x) dx = \begin{cases} 
\frac{\pi_0 c_1}{\gamma_1} \left( e^{\gamma_1/2} - e^{-\gamma_1/2} \right) & \text{if } \beta_1 \neq 0 \\
\pi_0 c_1 & \text{if } \beta_1 = 0 
\end{cases}
\]

\[
\pi_k = \int_{k-1}^{k} p(x) dx = \begin{cases} 
\pi_0 \left[ C_k + \frac{C_k - q_{k-1}}{1 - e^{-\gamma_k/2}} \left( \frac{\gamma_k}{\gamma_k/2 - 1} \right) \right] & \text{if } \beta_k \neq 0 \\
\pi_0 c_k & \text{if } \beta_k = 0 
\end{cases}
\]

\[
k = \int_{k-1}^{k} P_{m+1}(x) dx = \frac{\pi_0 \gamma_m C_1 \rho}{(1 - \rho) \left[ e^{\gamma_m (m-N)} - 1 \right]} \left[ 1 + \frac{e^{-\gamma_m (N-N)} - 1}{\gamma_m} \right]
\]

where \( m < k < N \)

Using normalizing condition

\[
\pi_0 + \int_0^{N} p(x) dx = 1
\]

The probability mass \( \pi_0 \) is given by

\[
\pi_0 = \begin{cases} 
\left( 1 - \frac{c_1 e^{-\gamma_1/2}}{\gamma_1} + \sum_{k=1}^{m} q_k \left( \frac{1}{\gamma_k} - \frac{1}{\gamma_{k+1}} \right) + \frac{C_k e^{\gamma_k/2 - q_k}}{\gamma_k} \right) + \frac{q_m \gamma_m}{\gamma_m} & \text{if } \beta_k \neq 0 \\
\frac{c_m \rho}{1 - \rho} \left( 1 + \frac{(N-m)\gamma_m}{\gamma_m (m-N) - 1} \right)^{-1} & \text{if } \beta_k = 0 
\end{cases}
\]

Using \( \pi_k \) \( (k = 1, 2, ..., N) \), we can obtain approximate formulae for the mean number of customers (including those in service) \( L \) and delay probability \( DLY \) as follows:
\[ L = \int_0^N x \ p(x) \, dx \]

\[ = \left\{ \begin{align*}
\pi_0 \left[ \frac{C_1 e^{-y_1}}{y_1} + \sum_{k=2}^{m} q_k \left( \frac{1}{y_k} - \frac{1}{y_{k+1}} \right) \left( k - \frac{1}{y_k} - \frac{1}{y_{k+1}} \right) + \frac{C_k e^{y_k/2} - q_k}{y_k/2 - 1} \right] (k - \frac{1}{y_k} - \frac{1}{2}) + \\
q_m \left( \frac{m}{y_m} \right) + \int_{N-1}^{N} x \ p(x) \, dx \quad \text{if } b_k \neq 0 \\
\pi_0 \left[ \frac{C_1}{2} + \sum_{k=2}^{m} C_k \left( k - \frac{2}{3} + \frac{1}{6} q_k \right) \right] \quad \text{if } b_k = 0
\end{align*} \]

Where

\[ \int_{N-1}^{N} x \ p(x) \, dx = \frac{y_m C_m \rho}{(1 - \rho)(e^{y_m(N-m)} - 1)} \left[ \frac{N^2 - m^2}{2} + \frac{m - N}{y_m} \right] + \frac{y_m C_m \rho}{(1 - \rho)} \left[ \frac{m}{y_m} + \frac{(1 + e)^{m+1} y_m}{y_m(1 - e^{y_m})} - \frac{1}{y_m^2} \right] \]

The delay probability (DLY) can be obtained by using the formula

\[ \text{DLY} = 1 - \sum_{k=0}^{m-1} \pi_k \]

4. Discussion

We have developed diffusion process associated with two reflecting boundaries, 0 and N for G/G/m/N model. There exits diffusion approximation for the same model applicable to FMS (see [28]) but suggested approximation is superior to earlier work. This work also improves the approximate results obtained by Sunaga et al. [15]. The interesting and important feature of the model investigated here is that it can be applied to flexible manufacturing system FMS, computers, telecommunication system etc.

References