Generalized Hyers-Ulam-Rassias Stability of a Reciprocal Type Functional Equation in Non-Archimedean Fields

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Abstract - In this paper, we obtain the general solution of a reciprocal type functional equation of the type

\[ f(x + y) = \frac{f\left(\frac{3x+2y}{5}\right) f\left(\frac{2x+3y}{5}\right)}{f\left(\frac{3x+2y}{5}\right) + f\left(\frac{2x+3y}{5}\right)} \]

And investigate its generalized Hyers-Ulam-Rassias stability in non-Archimedean fields. We also establish Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product-sum of powers of norms for the same equation.

I. INTRODUCTION


\[ f(mx + y) + f(mx - y) = 2f(x + y) + 2f(x - y) + 2(m^2 - 2)f(x) - 2f(y) \]

for any arbitrary but fixed real constant m with m\#0; m\#±1; m\#±\(\sqrt{2}\) using mixed product-sum of powers of norms. Several stability results have been recently obtained for various equations, also for mappings with more general domains and ranges (see [7], [8], [9], [11], [13], [18], [19], [20], [23]). Many research monographs are also available in functional equations, one can see ([1], [2], [3], [10], [15], [16], [17]).


\[ f(x + y) = \frac{f(x)f(y)}{f(x) + f(y)} \quad (1.1) \]

where \( f : X \rightarrow Y \) is a mapping on the spaces of non-zero real numbers. The reciprocal function \( g(x) = \frac{1}{x} \) is a solution of the functional equation (1.1).

K.Ravi, J.M. Rassias and B.V. Senthil Kumar [25] discussed the Ulam stability for the reciprocal functional equation in several variable of the form

\[ f\left(\sum_{i=1}^{m} \alpha_i x_i \right) = \frac{\prod_{i=1}^{m} f(x_i)}{\sum_{i=1}^{m} \alpha_i \left(\prod_{j=1,j\neq i}^{m} f(x_j)\right)} \quad (1.2) \]

for arbitrary but fixed real numbers \( (\alpha_1; \alpha_2; \ldots; \alpha_m) \neq (0; 0; \ldots; 0) \); so that \( 0 < \alpha_1 + \alpha_2 + \ldots + \alpha_m = \sum_{i=1}^{m} \alpha_i \neq 1 \) and \( f : X \rightarrow Y \) with X and Y are the spaces of non-zero real numbers.

Later, J.M. Rassias and et.al. [26] introduced the Reciprocal Difference Functional equation (1.3)

\[ f\left(\frac{x+y}{2}\right) - f(x+y) = \frac{f(x)f(y)}{f(x) + f(y)} \]

for arbitrary but fixed real numbers \( (\alpha_1; \alpha_2; \ldots; \alpha_m) \neq (0; 0; \ldots; 0) \); so that \( 0 < \alpha_1 + \alpha_2 + \ldots + \alpha_m = \sum_{i=1}^{m} \alpha_i \neq 1 \) and \( f : X \rightarrow Y \) with X and Y are the spaces of non-zero real numbers.
and the Reciprocal Adjoint Functional equation (1.4)

\[ f\left(\frac{x+y}{2}\right) + f(x+y) = \frac{3f(x)f(y)}{f(x) + f(y)} \]

and investigated the generalized Hyers-Ulam-Rassias stability of the equations (1.3) and (1.4).

A. Bodaghi and S.O. Kim [5] introduced and studies the Ulam-Gavruta-Rassias stability for the quadratic reciprocal functional mapping \( f : X \rightarrow Y \) satisfying the Rassias quadratic reciprocal functional equation

\[ f(2x+y) + f(2x-y) = \frac{2f(x)f(y)[4f(y) + f(x)]}{(4f(y) - f(x))^2} \]

(1.5)

The quadratic reciprocal function \( f(x) = \frac{c}{x^2} \) is a solution of the functional equation (1.5). Recently, A. Bodaghi and Y. Ebrahimdoost [6] generalized the equation (1.5) as (1.6)

\[ f((a+1)x) + f((a+1)x - ay) = \frac{2f(x)f(y)[(a+1)^2f(y) + a^2f(x)]}{((a+1)^2f(y) - a^2f(x))^2} \]

Where \( a \in \mathbb{Z} \) with \( a \neq 0 \) and established the generalized Hyers-Ulam-Rassias stability for the functional equation (1.6) in non-Archimedean fields.

K. Ravi et al [27] investigated the generalized Hyers-Ulam-Rassias stability of a reciprocal-quadratic functional equation of the form (1.7)

\[ r(x+2y) + r(2x+y) = \frac{r(x)r(y)[5r(x) + 5r(y) + 8\sqrt{r(x)r(y)}]}{[2r(x) + 2r(y) + 5\sqrt{r(x)r(y)}]^2} \]

In intuitionistic fuzzy normed spaces. In this paper we obtain the general solution of a reciprocal type functional equation of the type (1.8)

\[ f(x + y) = \frac{f\left(\frac{3x+2y}{5}\right) f\left(\frac{2x+3y}{5}\right)}{f\left(\frac{3x+2y}{5}\right) + f\left(\frac{2x+3y}{5}\right)} \]

And investigate the generalized Hyers-Ulam-Rassias stability of the equation (1.8) in non-Archimedean fields. We also establish Hyers-Ulam-Rassias stability, Ulam-Gavruta-Rassias stability and J.M. Rassias stability controlled by the mixed product sum of powers of norms for the equation (1.8).

II. PRELIMINARIES

A non-Archimedean field is a field \( A \) equipped with a function (valuation) \( | . | \) from \( A \) into \( (0,\infty) \) such that for all \( r,s \in A \)

\[
\begin{align*}
(i) & \quad |r| = 0 \text{ if and only if } r = 0 \\
(ii) & \quad |rs| = |r| |s| \text{ and} \\
(iii) & \quad |r + s| \leq \max \{ |r|, |s| \}.
\end{align*}
\]

Clearly \( |1| = |-1| = 1 \) and \( |n| \leq 1 \) for all \( n \in \mathbb{N} \).

We always assume, in addition, that \( | . | \) is non-trivial, i.e., there exists an \( a_0 \in A \) such that \( |a_0| \neq 0, 1 \).

An example of a non-Archimedean valuation is the mapping \( | . | \) taking every thing but 0 into 1 and \( |0| = 0 \). This valuation is called trivial.

Another example of a non-Archimedean valuation on a field \( A \) is the mapping.

Let \( p \) be a prime number. For any non-zero rational number \( x = p^m \frac{m}{n} \) in which \( m \) and \( n \) are coprime to the prime number \( p \). Consider the \( p \)-adic absolute value \( |x|_p = p^m \) on \( Q \). It is easy to check that \( | . | \) is a non-Archimedean norm on \( Q \). The completion of \( Q \) with respect to \( | . | \) which is denoted by \( Q_p \) is said to be the \( p \)-adic number field. Note that if \( p > 2 \), then \( |2^n| = 1 \) for all integers \( n \).

III. GENERAL SOLUTION OF EQUATION

Theorem 3.1. Let \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \) be a function. Then \( f \) satisfies (1.1) if and only if \( f \) satisfies (1.8). Hence (1.8) is also a reciprocal mapping whose solution is \( f(x) = \frac{c}{x} \).

Proof:
Let \( f \) satisfy (1.1). Replacing \((x,y)\) by \(\left(\frac{3x+2y}{5}, \frac{2x+3y}{5}\right)\) in (1.1), we arrive at (1.8).

Conversely, suppose \( f \) satisfy (1.8). Replacing \((x,y)\) by \((3x - 2y, 3y - 2x)\) in (1.8), we obtain (1.1). This completes the proof of Theorem 3.1.

**IV. GENERALIZED HYERS-ULAM STABILITY OF (1.8)**

In the following theorems and corollaries, we assume that \( A \) and \( B \) be a non-Archimedean field and a complete non-Archimedean field, respectively. From now on, for a non-Archimedean field \( A \), we put \( A^* - \{0\} \).

For convenience, let us define the difference operator \( D_f : A^* \times A^* \to B \) by

\[
D_f(x, y) = f(x + y) - \frac{f\left(\frac{3x+2y}{5}\right)}{f\left(\frac{2x+3y}{5}\right)} \cdot f\left(\frac{2x+3y}{5}\right)
\]

For all \( x, y \in A^* \). Theorem 4.1. Let \( \phi : A^* \times A^* \to B^* \) be a function such that

\[
\left| \frac{1}{2^n} f\left(\frac{x}{2^n}\right) \right| \leq \frac{1}{2^n} \phi\left(\frac{x}{2^n}, \frac{x}{2^n}\right)
\]

For all \( x, y \in A^* \). Suppose that \( f : A^* \to B \) is a mapping satisfying the inequality

\[
\left| D_f(x, y) \right| \leq \phi(x, y)
\]

(4.2)

For all \( x, y \in A^* \). Then there exists a unique reciprocal mapping \( r : A^* \to B \) such that (4.3)

\[
\left| f(x) - r(x) \right| \leq \max \left\{ \frac{1}{2^n} \left( \frac{x}{2^n+1}, \frac{1}{2} \right) : n \in \mathbb{N} \right\}
\]

For all \( x \in A^* \).

**Proof:** Replacing \((x, y)\) by \((x, x)\) in  (4.2), we get

\[
\left| f(2x) - \frac{1}{2} f(x) \right| \leq \phi(x, x)
\]

For all \( x \in A^* \). Now, replacing \( x \) by \( \frac{x}{2} \) in (4.4) we obtain (4.5)

\[
\left| f\left(\frac{x}{2}\right) - \frac{1}{2} f\left(\frac{x}{2}\right) \right| \leq \phi\left(\frac{x}{2}, \frac{x}{2}\right)
\]

For all \( x \in A^* \). Plugging \( x \) by \( \frac{x}{2^n} \) in (4.5) and multiplying by \( \frac{1}{2^n} \), we have (4.6)

For all \( x \in A^* \). Thus the sequence \( \left\{ \frac{1}{2^n} f\left(\frac{x}{2^n}\right) \right\} \) is Cauchy by (4.1) and (4.6).

Completeness of the non-Archimedean space \( B \) allows us to assume that there exists a mapping \( r \) so that (4.7)

\[
r(x) = \lim_{n \to \infty} \frac{1}{2^n} f\left(\frac{x}{2^n}\right)
\]

For each \( x \in A^* \) and non-negative integers \( n \), we have (4.8)

\[
\left| f\left(\frac{x}{2^n}\right) - f(x) \right| \leq \sum_{i=0}^{n-1} \left| \frac{1}{2^{i+1}} f\left(\frac{x}{2^{i+1}}\right) - \frac{1}{2^i} f\left(\frac{x}{2^i}\right) \right|
\]

\[
\leq \max \left\{ \frac{1}{2^{i+1}} f\left(\frac{x}{2^{i+1}}\right) - \frac{1}{2^i} f\left(\frac{x}{2^i}\right) : 0 \leq i < n \right\}
\]

\[
\leq \max \left\{ \frac{1}{2} \left( \frac{x}{2^{i+1}}, \frac{x}{2^i} \right) : 0 \leq i < n \right\}
\]

Applying (4.7) and letting \( n \to \infty \), we find that the inequality (4.3) holds. From (4.1), (4.2) and (4.7) we have for all \( x, y \in A^* \)

\[
\left| D_r(x, y) \right| = \lim_{n \to \infty} \frac{1}{2^n} \left| D_f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right|
\]

\[
\leq \lim_{n \to \infty} \frac{1}{2^n} \phi\left(\frac{x}{2^n}, \frac{x}{2^n}\right) = 0.
\]

Hence the mapping \( r \) satisfies (1.8). By Theorem 3.1, the mapping \( r \) is reciprocal. Now, let \( R : A^* \to B \) be another reciprocal mapping satisfying (4.3). Then we have

\[
\left| r(x) - R(x) \right| = \lim_{p \to \infty} \frac{1}{2^p} \left| f\left(\frac{x}{2^p}\right) - f\left(\frac{x}{2^p}\right) \right|
\]

\[
\leq \lim_{p \to \infty} \frac{1}{2^p} \max \left\{ \frac{1}{2} \left( \frac{x}{2^{p+1}}, \frac{x}{2^p} \right) : 0 \leq i < q \right\}
\]

\[
\leq \lim_{p \to \infty} \max \left\{ \frac{1}{2} \left( \frac{x}{2^{p+1}}, \frac{x}{2^p} \right) : 0 \leq i < q + p \right\}
\]

\[
= 0
\]

For all \( x \in A^* \), proving that \( r \) is unique, which completes the proof.

**Theorem 4.2.** Let \( \phi : A^* \times A^* \to B^* \) be a function such that (4.9)

\[
\left| 2^n \phi\left(2^n x, 2^ny\right) \right| = \left| 2^n \phi(x, y) \right|
\]

For all \( x \in A^* \). Thus the sequence \( \left\{ \frac{1}{2^n} f\left(\frac{x}{2^n}\right) \right\} \) is Cauchy by (4.1) and (4.6).
For all \( x, y \in A^* \). Suppose that \( f: A^* \to B \) is a mapping satisfying the inequality (4.2) for all \( x, y \in A^* \). Then there exists a unique reciprocal mapping \( r: A^* \to B \) such that (4.10)

\[
|f(x) - r(x)| \leq \max \left\{ |2^i \phi(2^i x, 2^i y) : i \in \mathbb{N} \cup \{0\} \right\}
\]

For all \( x \in A^* \).

**Proof:** Replacing \( (x, y) \) by \( (x, x) \) in (4.2) and multiplying by \( |2| \), we get (4.11)

\[
|2f(2x) - f(x)| \leq 2|\phi(x, x)|
\]

For all \( x \in A^* \). Switching \( x \) to \( 2^n x \) in (4.11) and multiplying by \( |2^n| \), we have (4.12)

\[
|2^n f(2^n x) - 2^{n+1} f(2^{n+1} x)| \leq 2^n |\phi(2^n x, 2^n y)|
\]

For all \( x \in A^* \). As \( n \to \infty \) in (4.12) and using (4.9), we see that the sequence \( \{2^n f(2^n x)\} \) is a Cauchy sequence. Since \( B \) is complete, this Cauchy sequence converges to a mapping \( r: A^* \to B \) defined by (4.13)

\[
r(x) = \lim_{n \to \infty} 2^n f(2^n x)
\]

For each \( x \in A^* \) and non-negative integers \( n \), we have (4.14)

\[
|2^n f(2^n x) - f(x)| = \sum_{i=0}^{n} 2^{-i} f(2^{-i} x) - 2 f(2^0 x)
\]

\[
\leq \max \left\{ |2^{-i} f(2^{-i} x) - 2 f(2^0 x) : 0 \leq i < n \right\}
\]

\[
\leq \max \left\{ 2^{-i} |\phi(2^{-i} x, 2^{-i} x) : 0 \leq i < n \right\}
\]

Applying (4.13) and letting \( n \to \infty \), we find that the inequality (4.10) holds. From (4.9), (4.2) and (4.13), we have for all \( x, y \in A^* \),

\[
|D_f(x, y)| = \lim_{n \to \infty} 2^n |D_f(2^n x, 2^n y)|
\]

\[
\leq \lim_{n \to \infty} 2^n |\phi(2^n x, 2^n y)| = 0.
\]

Hence the mapping \( r \) satisfies (1.8). By Theorem 3.1, the mapping \( r \) is reciprocal. Now, let \( R: A^* \to B \) be another reciprocal mapping satisfying (4.10). Then we have

\[
|R(x) - r(x)| = \lim_{p \to \infty} 2^p |R(p x) - r(p x)|
\]

\[
\leq \lim_{p \to \infty} 2^p \max \{|R(2^p x) - f(2^p x)|, |f(2^p x) - r(2^p x)|\}
\]

\[
\leq \lim_{p \to \infty} \lim_{q \to \infty} \max \{|2^i \phi(2^i x, 2^i y) : p \leq i \leq q + p\}
\]

\[
= 0.
\]

For all \( x \in A^* \), which proves that \( r \) is unique.

**Corollary 4.3.** For any fixed \( K_2 \geq 0 \) and \( \alpha \neq -1 \), if \( f: A^* \to B \) satisfies

\[
|D_f(x, y)| \leq k_1 \left( |x|^\alpha + |y|^\alpha \right)
\]

For all \( x, y \in A^* \), then there exists a unique reciprocal mapping \( r: A^* \to B \) satisfying (1.8) and

\[
|f(x) - r(x)| \leq \begin{cases} 
\frac{2k_1}{|x|^\alpha \alpha}, & \text{for } \alpha < -1 \\
\frac{4k_1}{|x|^\alpha}, & \text{for } \alpha > -1 
\end{cases}
\]

For every \( x \in A^* \).

**Proof:** The required results are obtained by choosing \( \phi(x, y) = k_2 \left( |x|^\beta + |y|^\beta \right) \), for all \( x, y \in A^* \) in Theorem 4.1 with \( \alpha < -1 \) and in Theorem 4.2 with \( \alpha > -1 \) and proceeding by similar arguments as in Theorems 4.1 and 4.2.

**Corollary 4.4.** Let \( f: A^* \to B \) be a mapping and let there exist real numbers \( a, b \): \( \alpha = a + b \neq -1 \). Let there exists \( k_2 \geq 0 \) such that

\[
|D_f(x, y)| \leq k_2 |x|^a |y|^b
\]

For all \( x, y \in A^* \). Then there exists a unique reciprocal mapping \( r: A^* \to B \) satisfying (1.8) and

\[
|f(x) - r(x)| \leq \begin{cases} 
\frac{k_2}{|x|^a}, & \text{for } \alpha < -1 \\
\frac{2k_2}{|x|^a}, & \text{for } \alpha > -1 
\end{cases}
\]

For every \( x \in A^* \).

**Proof:** Considering \( \phi(x, y) = k_2 |x|^a |y|^b \), for all \( x, y \in A^* \) in Theorem 4.1 with \( \alpha < -1 \) and in Theorem 4.2 with \( \alpha > -1 \), the proof of the corollary is complete.

**Corollary 4.5.** Let \( k_3 \geq 0 \) and \( \alpha \neq -1 \) be real numbers, and \( f: A^* \to B \) be a mapping satisfying the functional inequality

\[
|D_f(x, y)| \leq k_3 \left( |x|^\alpha |y|^\alpha + (|x|^\alpha + |y|^\alpha) \right)
\]

For all \( x, y \in A^* \). Then there exists a unique reciprocal mapping \( r: A^* \to B \) satisfying (1.8) and

\[
|f(x) - r(x)| \leq \begin{cases} 
\frac{2k_3}{|x|^\alpha}, & \text{for } \alpha < -1 \\
\frac{12k_3}{|x|^\alpha}, & \text{for } \alpha > -1 
\end{cases}
\]

For every \( x \in A^* \).

**Proof:** The proof follows immediately by taking \( \phi(x, y) = \left( |x|^\alpha |y|^\alpha \right)^{\frac{a}{2}} \) in Theorem 4.1 with \( \alpha < -1 \) and in Theorem 4.2 with \( \alpha > -1 \).

**References**


