Common Fixed Point Of Compatible Maps in D-Metric Spaces

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Abstract- In this paper, a transposition of these notions is being made for 4-tuples A, B, C, D of self maps of a D-metric space (X, D). Under suitable contractive conditions, some common fixed point theorems involving such maps have been proved by several authors.

Keywords- D-metric space, D-weakly compatible mappings

I INTRODUCTION

Dhage [1] introduced the concept of D-metric space. The generalized metric D is a slight enlargement of the 2-metric concept. He also gave a generalization of Banach’s contraction principle for such structures; cf Rhoades [8]. Further Dhage [2], Raju, Rao and Rao [7], Veerapandi and Rao [12], Sharma and Dawar [10] and many others have used this concept in fixed point frame work.

In 1976, Jungck [4] proved a common fixed point theorem for commuting mappings. Sessa [9] defined a generalization of commutativity, which is called weak commutativity. Further Jungck [5], [6] introduced more generalized commutativity so called compatibility, which is more general than that of weak commutativity.

II PRELIMINARIES

Definition 1 [1]. Let X be a non-empty set. A generalized metric (or D-metric) on X is a function D from X× X× X to R+ that satisfies the following conditions:

(D-1) D(x, y, z) = 0 if and only if x = y = z
(Sufficiency),

(D-2) D(x, y, z) = D(y, x, z) = ...
(Symmetry),

(D-3) D(x,y,z) ≤ D(x,y,a) + D(x,a,z) + D(a,y,z) for all x,y,z in X.
(Rectangle inequality)

The pair (X, D) is then called a D-metric space. As immediate examples of such functions we quote, cf. Dhage [2]:

(a) D(x,y,z) = max{d(x,y),d(y,z),d(z,x)} and
(b) D(x,y,z) = d(x,y) + d(y,z) + d(z,x).

Here, d is an ordinary metric on X. This notion is closely related to the one of the 2-metric, introduced by Gähler [3] as follows:

Definition 2. Let X be a non-empty set. A 2- metric on X is a function d from X× X× X to R+ which satisfies the following conditions:

(d-1) for every distinct pair of x, y ∈ X there is a z ∈ X such that

d (x, y, z) ≠ 0,

(d-2) d(x, y, z) = 0 if and only if any two of the triplet x, y, z are equal,

(d-3) d(x, y, z) = d(y, x, z) = ...
(Symmetry),

(d-4) d(x,y,z) ≤ d(x,y,a) + d(x,a,z) + d(a,y,z) for all x,y,z,a in X
(Rectangle inequality)

Geometrically a 2-metric $d(x, y, z)$ represents the area, while the D-metric $D(x, y, z)$ represents the perimeter of a triangle whose vertices are $x, y$ and $z$.

Definition 3 [2]. A sequence $\{x_n\}$ of points in a D-metric space $(X, D)$ is said to be D-convergent to the point $x \in X$ if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$D(x_n, x_m, x) < \varepsilon,$$

and convergent, if it converges to some point $x$ of $X$.

Definition 4. [1] A sequence $\{x_n\}$ of points in a D-metric space $(X, D)$ is said to be D-Cauchy if for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$D(x_n, x_m, x_p) < \varepsilon,$$

for all $m, p > n_0$.

Definition 5. [1] A D-metric space $(X, D)$ is said to be complete if every D-Cauchy sequence in $X$ converges (to a point in $X$).

Definition 6. Two self mappings $A$ and $B$ of a D-metric space $X$ are said to be D-weakly commuting if

$$D(ABx, BAy, z) \leq D(Ax, By, z),$$

where $z = Ax$ (or $Bx$), for all $x \in X$.

Definition 7. Two self mappings $A$ and $B$ of a D-metric space $X$ are said to be D-compatible if

$$\lim_{n \to \infty} D(ABx_n, BAx_n, z) = 0,$$

where $z = ABx_n$ (or $BAx_n$)

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = y$, for some $y$ in $X$.

Clearly, commutativity implies D-weak commutativity and D-weak commutativity implies D-compatibility; but implication is not always reversible, see [1].

Definition 8. Two self mappings $A$ and $B$ of a D-metric space $X$ are said to be D-weakly compatible if they commute at their coincidence points.

MAIN RESULTS

Let $\Phi$ be the family of all continuous non-decreasing functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\phi(t) < t$, for $t > 0$ and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t \in \mathbb{R}_+$.

Singh and Sharma [11] proved the following:

Theorem A: Let $A$, $B$, $F$ and $G$ be self – mappings of a complete and bounded D-metric space $(X, D)$ satisfying:

(i) $A(X) \subseteq G(X)$ and $B(X) \subseteq F(X),$

(ii) One of $A$, $B$, $F$ or $G$ is continuous,

(iii) $(A, F)$ and $(B, G)$ are D-compatible pairs of mappings,

(iv) $D(ABx, BAy, z) \leq \phi[\max\{D(Fx,Gy,z), \alpha D(Fx,By,z), \alpha D(Gy,Ax,z)\}]$

for all $x, y, z \in X$, where $\phi \in \Phi$ and $0 < \alpha \leq 1/3$.

Then $A$, $B$, $F$ and $G$ have a unique common fixed point in $X$.

We prove Theorem A without taking any function continuous and replacing the condition of D-compatible mappings by D-weakly compatible mappings. We prove the following.

Theorem 1. Let $A$, $B$, $F$ and $G$ be self – mappings of a complete and bounded D-metric space $(X, D)$ satisfying conditions (i) and (iv) for all $x, y, z \in X$, where $\phi \in \Phi$ and $0 < \alpha \leq 1/3$ and
(1) (A, F) and (B, G) are D-weakly compatible pairs of mappings.

Then A, B, F and G have a unique common fixed point in X.

Proof. By assumption (i), since \( A(X) \subseteq G(X) \), for an arbitrary \( x_0 \in X \) there exists a point \( x_1 \in X \) such that \( A x_0 = G x_1 \). Since \( B(X) \subseteq F(X) \), for this point \( x_1 \) we can choose a point \( x_2 \in X \) such that \( B x_1 = F x_2 \). Continuing this way, we can construct a sequence \( \{y_n\} \) in X such that

\[
y_{2n} = F x_{2n} = B x_{2n-1} \quad \text{and} \quad y_{2n+1} = G x_{2n+1} = A x_{2n},
\]

for every \( n = 1, 2, \ldots \).

Since the conditions (i) and (iv) of our theorem are similar to that of Theorem A of Singh and Sharma [11] therefore, we conclude that \( \{y_n\} \) is a D-Cauchy sequence in X. By completeness of X, \( \{y_n\} \) and also its subsequences \( \{A x_{2n}\} \), \( \{B x_{2n-1}\} \) and \( \{G x_{2n+1}\} \) converge to \( u \) in X.

Since \( B(X) \subseteq F(X) \), there exists a point \( q \in X \), such that \( u = F q \).

Using (iv), we have

\[
D(u, Bp, u) = D(Aq, Bp, u) \leq \phi[ \max\{D(u,Gp,u), \alpha D(Gp,u,u)\}]
\]

Taking the limit \( n \rightarrow \infty \), we get

\[
D(u, Bp, u) \leq \phi[ \max\{D(u,Gp,u), \alpha D(Gp,u,u)\}]
\]

Hence \( Aq = Fq, \ Bp = Gp = u \).

Since the maps A and F are D-weakly compatible, then \( AFq = FAq \) i.e.

\[
Au = Fu. \quad \text{Now we show that} \ u \ \text{is a common fixed point of A.}
\]

By (iv), we have

\[
D(Au,Bx_{2n+1},u) \leq \phi[ \max\{D(Fu,Gx_{2n+1},u), \alpha D(Fu,Bx_{2n+1},u)\}]
\]

Using (iv), we have

\[
D(u,w,u) = D(Au,Bw,u) \leq \phi[ \max\{D(Fu,Gw,u), \alpha D(Fu,Bw,u)\}]
\]
\[ < D (u, w, u), \]

Which yields \( u = w \). This completes the proof.

**Corollary 1:** Let \( A \) and \( F \) be self-mappings of a complete and bounded \( D \)-metric space \( (X, D) \) satisfying:

\[ \text{(2) } A(X) \subseteq F(X), \]
\[ \text{(3) } (A, F) \text{ is } D-\text{weakly compatible pair of mappings}, \]
\[ \text{(4) } D(Ax, Ay, z) \leq \phi[ \text{max} \{ D(Fx, Fy, z), \alpha D(Fx, Ay, z), \]
\[ \alpha D(Fy, Ax, z) \}] \]

for all \( x, y, z \in X \), where \( \phi \in \Phi \) and \( 0 < \alpha \leq 1/3 \).

Then \( A \) and \( F \) have a unique common fixed point in \( X \).

Finally, when \( F = I \) (identity mapping on \( X \)) in Corollary 1, we get

**Corollary 2:** Let \( A \) be a self mapping of a complete and bounded \( D \)-metric space \( (X, D) \) satisfying:

\[ \text{(5) } D(Ax, Ay, z) \leq \phi[ \text{max} \{ D(x, y, z), \alpha D(x, Ay, z), \]
\[ \alpha D(y, Ax, z) \}] \]

for all \( x, y, z \in X \), where \( \phi \in \Phi \) and \( 0 < \alpha \leq 1/3 \).

Then \( A \) has a unique fixed point in \( X \).

**Conclusions**

We prove Theorem without taking any function continuous and replacing the condition of \( D \)-compatible mappings by \( D \)-weakly compatible mappings

**References:**


