

Size and Topological Optimization of Cantilever Beam

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Abstract— Optimization is the process of selecting the best possible results among many possible results under given circumstances. Mathematically it is finding the global maxima or minima of a function. This paper concerns with the optimization of a cantilever beam. The governing idea is to reduce the mass of the beam for a given strength. The cross section under consideration is a rectangular one. A Graphical Approach has been adopted to plot SOM equations on Matlab to obtain a common acceptable domain. A traditional method, Johnson’s method was considered to account for fatigue strength of the material and obtain optimum dimensions. Finally COMSOL software was used to topologically optimize the beam.

Keywords— Structural Optimization, Cantilever Beam, Johnson’s method, topological optimization, COMSOL.

I. INTRODUCTION

Optimization is a process of selecting the best possible results among many possible results under given circumstances. Mathematically it is finding the global maxima or minima of a function. Structural Optimization is applying optimization to conventional design process to get best possible value for design variables based on certain criteria. In the simplest case it is the process of selection of materials and geometry which satisfy specified and implied functional requirements while remaining within the confines of inherently unavoidable limitations.

In this time of advancement and development, everything comes at a cost. Today, all organizations look at reducing liabilities and improving their profit margin. The concept of ‘A penny saved is a penny earned’ is the guideline that serves validly today. Time is also weighed in gold today. Thereby it requires a miser’s attitude to achieve results by keeping a tab on both of the above parameters which remain vital pillars behind any engineering work. Hence arrives the need for optimization.

This paper explores various tools used for optimization. All methods are carried out on a cantilever beam of rectangular cross-section.

A few popular optimization techniques are as follows:

Table I
Optimization Techniques

Traditional Optimization Techniques	Modern Optimization Techniques
Calculus Methods	Genetic Algorithm
Gradient Based	Simulated Annealing

Geometric Programming	Ant Colony Optimization
Linear Programming	Particle Swarm Optimization
Quadratic Programming	Neural Networks
Dynamic Programming	Fuzzy Logic
Stochastic Programming	
Multi-Objective Programming	

II. BEAMS

A beam is a long structural member with a relatively small cross-section dimension and is generally subjected to transverse forces. Typical applications of beams are:

- Support floors and ceilings of building
- Resists bending due to forces on an aircraft beam
- Bridges

A. Stresses in Beams

Internally beams experience compressive, tensile and shear stress as a result of loads applied on them.

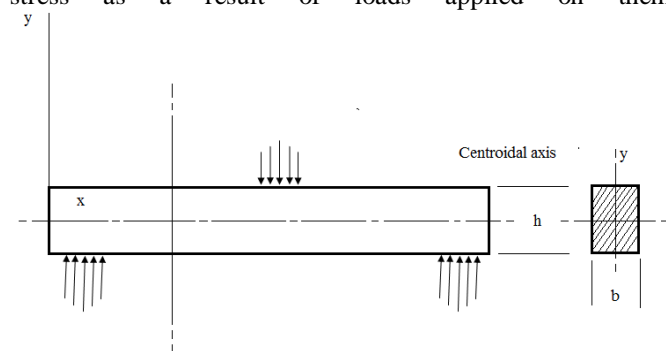


Fig.1 Simply supported beam showing cross-section

The primary tool for structural analysis is the Euler-Bernoulli Equation^[5]. Other mathematical methods for determining deflection of beams include “method of virtual work”^[6] and “slope deflection method”.

Mathematical methods for determining the beam forces (internal forces of the beam that are imposed on the beam support) include the “moment distribution method”^[7], the force or “flexibility method”^[8] and the “direct stiffness method”^[9].

The fundamental equations used for Graphical and Johnson’s Methods are^[10]:

- Flexure Equation
- Basic Deflection Equations

Deflection Equation for Cantilever Beam

III. GRAPHICAL METHOD

This method can be employed for single-objective constrained problems involving one or two variables. It's the simplest type of optimization technique and gives results which can be used as a reasonably accurate initial-guess for more powerful algorithms.

In this method all the design equations are plotted simultaneously. The intersection of these plots gives the optimal design using this method.

The design of the cantilever beam of rectangular cross-section, subjected to a point end-load falls into this category.

A. Formulation of Design Equations

The width (x_2) and height (x_1) of the beam are the design variables.

Input variables are force applied (F), length of the beam (L), Young's Modulus I, Yield Stress (σ_b), Factor of Safety (FOS), max deflection (δ_{max}) and the limits on the height and width of the beam.

The equations derived by author#1 are as follows:

- **Design Vector** (X)=(d,b), (x_1, x_2)
- **Objective Function**, $f(x) = \rho \times L \times x_1 \times x_2$
- **Subsidiary Design Equation 1**
 - From Flexure Equation, $\frac{M}{I} = \frac{\sigma_b}{y} = \frac{E}{R}$
 - $\sigma_b = \frac{6FL}{x_1^2 \cdot x_2}$
 - $\tau_{induced} = \frac{\sigma_b}{2} = \frac{3FL}{x_1^2 x_2}$
 - According to max shear stress theory
 - $\tau_{max} = \frac{\sigma_y}{2.FOS}$

Hence $\frac{3FL}{x_1^2 \cdot x_2} < \frac{\sigma_y}{2.FOS}$

○ $\frac{6FL * FOS}{\sigma_y} - x_1^2 \cdot x_2 < 0$

- **Subsidiary Design Equation 2**
 - Tip Deflection of Cantilever Beam,
 - $\delta_{tip} = \frac{FL^3}{3EI}$
 - $\delta_{tip} < \delta_{max}$
 - $(\frac{4FL^3}{E}) * (\frac{1}{x_1^3 \cdot x_2}) < \delta_{max}$
 - $(\frac{4FL^3}{E \cdot \delta_{max}} - x_1^3 \cdot x_2) < 0$

The Matlab software used to create a code which will plot the equations is given below:

```
clear all;
close all;
clc;
FOS= input(' Enter the value for FOS= ');
F= input(' Enter Force= ');
L= input(' Length= ');
E= input(' Enter Youngs Modulus for Material= ');
sigma_y= input(' Yield Stress of the Material= ');
def= input(' Maximum allowable deflection= ');
a= input(' Minimum depth of the beam= ');
b= input(' Maximum depth of the beam= ');
darshan1= 6*F*(L^1.5)*FOS/sigma_y;
x1= a:0.001:b;
[r,c]=size(x1);
x2=x1;
for i=1:1:c
    x2(1,i)=darshan1/x1(1,i)^2;
end
darshan2=4*F*(L^3)/E*def;
x3=x1;
for i=1:1:c
    x3(1,i)=darshan2/x1(1,i)^3;
end
plot(x1,x2,x1,x3);
```

The following input values were fed into the Matlab Code

Table II
Graphical Method Input Values

Parameter	Value	Units
L	1000	Mm
F	2000	N
E	200e3	Mpa
σ_y	56	Mpa
Minimum b	14	mm
Maximum b	200	mm

B. Results of Graphical Method

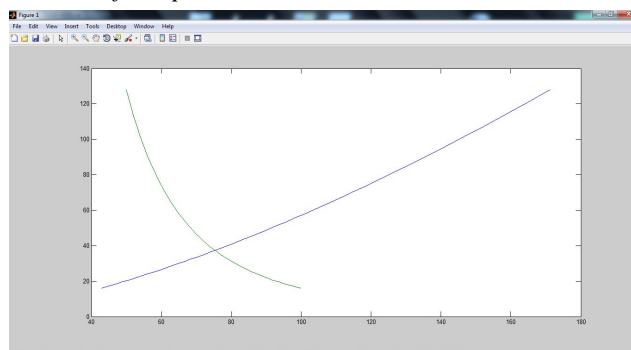


Fig.2 Plot of Depth vs Width of the two design equations

By this method, the optimum dimensions of the beam are width 38mm and height 76mm.

IV. TOPOLOGICAL OPTIMIZATION USING COMSOL

A. Introduction to Topological Optimization^[3]

Topology optimization is a mathematical approach that optimizes material layout within a given design space, for a given set of loads and boundary conditions such that the resulting layout meets a prescribed set of performance targets.

Topology means the way parts of an object are connected to each other. It is a more general approach than shape and size optimization. No prior assumptions of the shape are needed to carry out the optimization.

Topology optimization treats the material distribution, as defined by the finite element mesh, as the design variables. Each element in the finite element mesh has an additional scalar variable that is used to track the presence of the material within that element. The set of these scalar variables, one per element, are the design variables to be changed in an attempt to improve the cost function, while satisfying the constraints.

Despite being a powerful optimization method and easy to set up, Topology Optimization has many issues:

1. The optimal design is often not a candidate for manufacture
2. The resultant design has some dependence on the mesh size
3. Difficult to incorporate geometric constraints
4. Can result in checker-board pattern

Therefore, topology optimization should be used early in the design process as a guide for the overall structure of the system. It becomes more difficult to use efficiently when the design is already quite rigid and only minor modifications to the design are possible.

B. Topology Optimization Design Steps

In general, we distinguish the following steps:

Pre-processing of geometry and loading

- Choose a suitable reference domain (ground structure)
- Choose the part to be designed (the solid domain and voids)
- Construct FE mesh for the ground structure
- Construct FE spaces for displacements and ρ (design variables)

Optimization

- Compute the optimal distribution over the reference domain of the design variable ρ
- Displacement base FE method analysis
- Optimality update criteria scheme for the density

Post-processing of results

- CAD representation

C. Methodology^[4]

The model used is the *SIMPS* (solid isotropic material with penalization) method to determine the optimal topology. In this model, this means using a model for Young's modulus E such that $E = \mu(x)^p E_0$, where $0 < \mu(x) \leq 1$. The exponent p is a parameter such that $p = 1$ corresponds to ignoring the binary nature of μ whereas higher values of p yields more binary structures. In the model, p is set to 5.

The objective functional for the optimization, which defines the criterion for optimality, is the *strain energy* in this model. The control variable is ρ_{design} , which is constrained, using a pointwise constraint, to a value between 10^{-4} and one (1) and, using an integral constraint to be smaller than the area of the design domain (the entire beam geometry) times an area fraction, which is a parameter *area_frac* set to 0.5.

An integral inequality constraint which provides regularization is set up to reduce the dependency on the mesh size and prevent checker-board pattern solution. Structural Steel is used in the model.



Fig.3 Geometry used in COMSOL



Fig.4 Extra-fine mesh used for model

The optimization solver used is the SNOPT solver. The QP solver used is Cholesky. A direct, fully-coupled stationary study is carried out.

D. Results

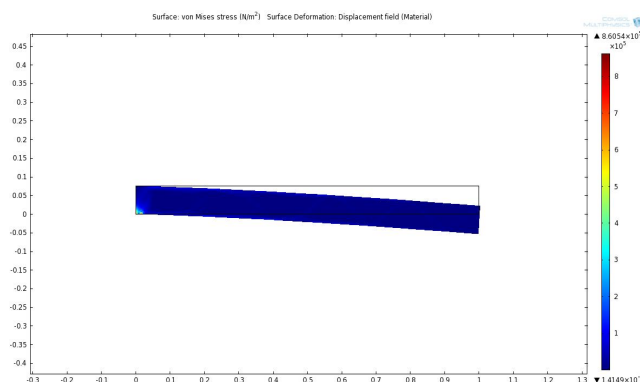


Fig 5 2D plot showing deflection and Von-Mises Stress

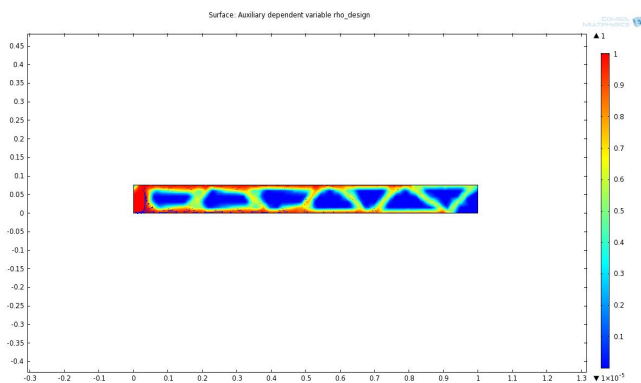


Fig.6 2D plot showing density distribution

From the above figure it is clear that the regions seen in red are highly stressed while the region show in blue has very less stress.

Hence material from the blue region can be scooped out. This results is a truss-like lower weight structure which can withstand all the forces.

V. JOHNSON’S METHOD

A. Introduction

In this particular methodology, a beam for simplicity purpose, is considered to function in a device as spring member. The device is to be manufactured in mass quantity, and hence the engineer is required to design its elements on the basis of minimum cost. The flat spring is to have a force gradient of a specified value K_1 , a specified length of L , constant force of F and a predetermined factor of safety N . The spring materials generally preferred are phosphor bronze, beryllium copper, or spring steel whose mechanical properties are given in Table 3.2. An assumption made here is that the spring is manufactured with appropriate production methods in quantities sufficiently large that the tooling, labor and fixed costs will be the same for any feasible design. Thus, the problem of optimum design is to select the optimum material and the optimum values for the geometrical parameters b and h such that the material cost for the manufactured spring is minimized.

The primary design equation expressing the material cost for the manufactured spring is

$$C=cW=cwV=cwLbh$$

$$C=cwLbh$$

where, C is the material cost for the manufactured spring which is to be minimised by optimum design, c is the unit material cost in US\$ per unit weight, w is the weight density of the material in N/m^3 , L is the specified spring length in m, and b and h are geometrical parameters whose optimum values are determined by the design procedure. Deflection of a cantilever beam with end load is given as

$$Y = \left(\frac{1}{6}\right) \times \left(\frac{F}{EI}\right) \times (x^3 - 3L^2x + 2L^3)$$

$$MaxY = \frac{FL^3}{3EI} \quad I = \frac{bh^3}{12} \quad \text{At } x=L$$

Hence,

$$MaxY = \frac{4FL^3}{Eb^3h^3}$$

$$K_1 = \frac{F}{MaxY} = \frac{F}{\left(\frac{4FL^3}{Eb^3h^3}\right)}$$

$$K_1 = \left(\frac{EBH^3}{4L^3}\right)$$

Above equation is called subsidiary design equation. E is the modulus of elasticity of the material and the other terms are as specified before.

Table III
Material Properties^[12]

Material	C(cost per unit weight)(US\$/tonne*g)	W(Weight Density)(kg*/m ³)	E(Yo ung's Modulus)(G Pa)	S _e (Fatigue Strength)(MPa)	S _{ey} (Fatigue Yield Strength)(MPa)	P=(S _e /S _{ey})
Phosphor Bronze	1000-2100	89000	111	172	180	0.95556
Beryllium Copper	200-700	81000-82500	124.2	210-805	410-1400	0.5608

A second subsidiary design equation is written to express the significant stresses, corresponding to the selected theory of failure. Here, the reasonable and generally conservative maximum shearing stress criterion for fatigue failure is taken. Thereby, $\tau_{max} = \frac{Mh}{4I}$ or $\tau_{max} = \frac{3F}{2bh}$ is taken depending of which governs. Assuming,

$$\frac{Mh}{4I} \geq \frac{3F}{2bh}$$

$$\frac{FLh}{4\frac{bh^3}{12}} \geq \frac{3F}{2bh}$$

i.e. $L \geq \frac{h}{2}$

The above relation is valid for any reasonable flat spring design. Hence, it is concluded with assurance that it is a reasonable assumption. Thus,

$$\tau_{max} = \frac{Mh}{4I} = \frac{FLh}{4\frac{bh^3}{12}}$$

$$\tau_{max} = \frac{3FL}{bh^3}$$

In the above equation, τ_{max} is the greatest shearing stress in the beam at any time and others are as specified as before.

From fatigue failure criteria, we obtain a limit equation. The greatest shearing stress in the bar at any time is given by τ_{max} , and according to the variation of force as considered, the mean value for that shearing stress is $0.5 \tau_{max}$. Hence,

$$\tau_{\max} - \frac{(1-p)}{2} \tau_{\max} \leq \frac{S_e}{2N}$$

$$\tau_{\max} \leq \frac{S_e}{(1+p)N}$$

where, S_e is the fatigue strength of the material for the desired spring life determined by a reversed bending type of standard test, N is factor of safety, and p is the ratio $S_e/(S_y)$, where (S_y) is the published yield strength of the material from a simple tensile test.

Another limit equation that is taken, so as to consider the space restriction existing on the feasible values of the geometrical parameter b is,

$$b_{\max} \geq b \geq b_{\min}$$

where, b_{\max} and b_{\min} define the range of feasible values for the geometrical parameter b .

B. Formulation of Objective Equations^[11]

In this case redundant specifications exist as it is cumbersome to obtain an equation involving all limit and subsidiary equations at the same time. Thus temporarily ignore the limit equation involving geometric parameters. Eliminating the independent geometric parameters, we can write

$$\left(\frac{K_1}{(\tau_{\max})^2} \right) = \frac{Ebh^3}{4L^3} \frac{b^2h^4}{9F^2L^2}$$

Rearranging,

$$bh = \frac{9EF^2}{4LK_1\tau_{\max}^2}$$

Substituting above in the primary design equation, $C=cwLbh$, we get

$$C = cwL \frac{9EF^2}{4LK_1\tau_{\max}^2}$$

Rearranging it, we have

$$C_{\min} = \frac{9}{4} \frac{F^2 N^2}{K_1} \frac{(cw)E(1+p)^2}{S_e^2}$$

This equation represents the minimum cost parameter which would otherwise have been the final design equation if there were no geometrical limit equation. Now to consider the limit equation involving geometric parameter, we can write from

$$K_1 = \frac{Ebh^3}{4L^3} \text{ and } \tau_{\max} = \frac{3FL}{bh^3},$$

$$h^6 = \frac{16K_1^2 L^6}{E^2 b^2}$$

and

$$h^6 = \frac{27F^3 L^3}{\tau_{\max}^3 b^3}$$

respectively.

Equating above equations and rearranging we obtain,

$$\left(\frac{1}{\tau_{\max}} \right)^2 = \left(\frac{16}{27} \right)^{\frac{2}{3}} \frac{K_1^{\frac{4}{3}} L^2 b^{\frac{2}{3}}}{E^{\frac{4}{3}} F^2}$$

Substituting above equation in C_{\min} we get

$$C = \frac{2^{\frac{2}{3}} K_1^{\frac{1}{3}} L^2 (cw)}{E^{\frac{1}{3}}} b^{\frac{2}{3}}$$

This is the final design equation in which cost is a function of b . On similar lines, we have

$$b = \frac{4K_1 L^3}{Eh^3}$$

from $K_1 = \frac{Ebh^3}{4L^3}$ and

$$b = \frac{3FL}{h^2 \times \tau_{\max}}$$

$$\text{from } \tau_{\max} = \frac{3FL}{bh^3}$$

Equating above equations and rearranging we obtain,

$$\left(\frac{1}{\tau_{\max}} \right)^2 = \left(\frac{16}{9} \right)^{\frac{2}{3}} \frac{K_1^2 L^4}{E^2 h^2 F^2}$$

Substituting above equation in C_{\min} , we get

$$C = \frac{4K_1 L^4 (cw)}{Eh^2}$$

This is the final design equation in which the cost is a function of h .

C. Matlab Code

```
E=input('Enter Young modulus of material in Pa :');
L=input('Enter length of Beam in m :');
c=input('Enter the cost per unit weight of the material in US$/kg*g :');
w=input('Enter the weight density of the material in N/m^3 :');
S=input('Enter the fatigue strength of material in Pa :');
Y=input('Enter the yield strength of the material in Pa :');
N=input('Enter the factor of safety :');
F=input('Enter the value of force on the free end of the beam in N :');
K1=input('Enter the value of force gradient in N/m :');
Bmin=input('Enter the min width of the beam cross section in m :');
Bmax=input('Enter the max width of the beam cross section in m :');
P=(S/Y)
Cmin=((9/4)*((F^2)*(N^2)/K1)*((c*w*E*(1+P)^2)/S^2))
A=(2^(2/3)*(K1^(1/3)*L^2)*((c*w)/E^(1/3)))
B=(4*K1*L^4*((c*w)/E))
b=0:.01:2;
h=0:.01:2;
cost1=A*(b.^(2/3));
cost2=B*(h.^(-2));
subplot(1,2,1)
loglog(b,cost1,'b-',Bmin,cost1,'r--',Bmax,cost1,'r--',b,cmin,'g-')
title('MATERIAL NAME//')
xlabel('breadth of beam (b) in m')
ylabel('cost of beam in US$')
gtext('Cmin')
gtext('Bmin')
gtext('Bmax')
gtext('Slope = 2/3')
subplot(1,2,2)
loglog(h,cost2,'b-',h,cmin,'g-')
title('MATERIAL NAME//')
xlabel('depth of beam (h) in m')
ylabel('cost of beam in US$')
gtext('cmin')
gtext('Slope = - 2')
```

D. Results

Analysis is done by looking at the variation of C versus b and C versus h for material phosphor bronze as a case study. It is done so on a log-log sketch by plotting the main design

equations with the geometrical limit equations and the minimum cost equation. The common region of the lines give the range of geometrical values that are optimized. Thus taking the log of (3.11) and (3.12) on both sides, we get

$$\log(C) = \log \left[2^{\frac{2}{3}} \times K_1^{\frac{1}{3}} \times L^2 \times \frac{cw}{E^{\frac{1}{3}}} \right] + \frac{2}{3} \log(b)$$

where,

$$\log \left\{ \frac{2^{\frac{2}{3}} * K_1^{\frac{1}{3}} * L^2 (cw)}{E^{\frac{1}{3}}} \right\}$$

is a constant. Hence it represents a straight line with slope (2/3). Also,

$$\log C = \log \left(\frac{4K_1 L^4 (cw)}{E} \right) - 2 \log h$$

where, $\log \left(\frac{4K_1 L^4 (cw)}{E} \right)$ is a constant. Hence it represents a straight line with slope (-2).

Subsequent coding is done on Matlab for the material phosphor bronze with parameters taken in standard units as follows,

$E=111*(10^9)$, $L=25$
 $c=1$, $w=89*(10^3)$, $S_e=172*(10^6)$, $(S_e)y=180*(10^6)$
 $N=1.5$, $F=500$, $K_1=100$
 $b_{min}=0.025$, $b_{max}=0.5$

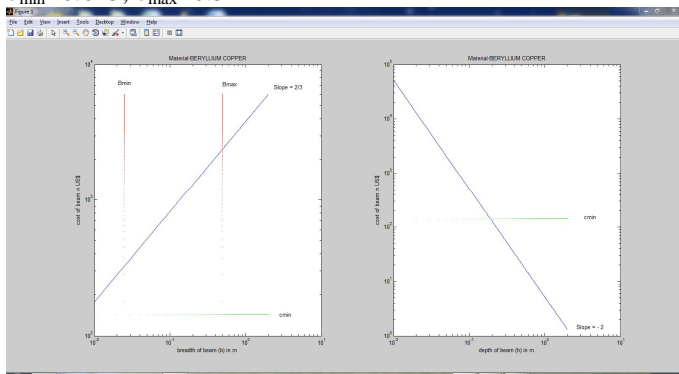


Fig.7 Plot of cost vs geometrical parameters

VI. CONCLUSIONS

These methods were applied to a general problem, pertinent to a cantilever beam of rectangular cross-section and the results were analysed. It was found that:

Graphical method gave the breadth and height of a rectangular cantilever beam based on basic strength of material equations

Johnson’s method furthered the graphical method and took into consideration Fatigue strength of material and spring stiffness while minimizing the cost.

Comsol software gave us a cantilever beam which is topologically optimized having a truss-like structure which has maximum stiffness and volume about half the initial volume of the beam.

This paper attempted to provide a bird-eye view of the structural optimization, with a special emphasis on its relation to optimization of cantilever beam.

The future research in this area will continue to address:

Technologies for structural optimization in conceptual design: method for generating innovative design concepts through optimization.

Technologies for large-scale structural optimization: efficient approximation methods for large-scale, nonlinear, and robust/reliability optimization problems.

ACKNOWLEDGMENT

The authors wish to acknowledge Prof. H.K.Rangavittal, Associate Professor, Mechanical Dept., B.M.S. College of Engineering, Bangalore, for initially introducing us to the realm of design optimization. His knowledge and experience in this field was of immense help and has played a major role in the completion of this paper.

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