Semi Quadratic Analytic Method for Neumann Localized Boundary-Domain Integral Equations

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ABSTRACT

Some of the integrals in a Localized Boundary Domain Integral Equations (LBDIEs) encounter problem of having singularity expression in part of its integrals' kernel. The singularities are of logarithmic singularity and r^{-2} singularity, where r is the distance between the field point y and source point x. This paper propose a method named as quadratic semi analytic method to get rid the influence of r^{-2} singularity which is part of one integration kernel in the Neumann LBDIEs. The derivation of this method is important to invalidate the singularity influence in order to get high accuracy in numerical experiments. The key idea is represent the difficult boundary integral with r^{-2} singularity of $T_x P_\tau(x,y)$ as two integrals. The first integral is without the coefficient variable $b(x)$ can be calculated exactly whereas the second integral with the coefficient variable $b(x)$ is to be computed numerically. This idea is possible to overcome the influence of r^{-2} from the fact that the analytic solution moves faster than its numerical solution. As for implication, the derivation of this quadratic semi analytic method helps to invalidate the singularity influence of r^{-2} whenever x is near to y without having to calculate the integral exactly.

Keywords: *Localized boundary-domain integral equations, partial differential equations, Neumann problem, semi quadratic analytic method, singularity.*

I. INTRODUCTION

Partial Differential Equations (PDEs) that include Laplace's equation, Helholmz's equation, heat equation, convection-diffusion equation, wave equation can be met in mathematical physics and engineering. See e.g. [1]. A remarkable revolution of computer's technologies has boosted the development of the numerical methods in solving PDEs. Among the most popular numerical methods to solve Boundary Value Problems (BVPs) for the PDEs are Finite Element Method (FEM), Finite Difference Method (FDM) and Boundary Element Method (BEM). When comparing BEM with both alternative methods i.e. FEM and FDM, [1] stated that BEM is vastly superior in efficiency and accuracy. The dimensionality of both alternative methods preserves from the dimensionality of the original physical problem. On the other hand, the dimensionality of the BEM is reduced by one with respect the dimensionality of the original physical problem. See e.g. [1]-[2].

However, the reduction of the dimensionality is mostly applicable for the BVPs for PDEs with constant coefficient. This is due to the availability of the fundamental solution for PDEs with constant coefficients. The idea is to present the solution in terms of boundary distributions of fundamental solution. The fundamental solution arise from the older idea of a Green's function. unfortunately, a fundamental solution is not always accessible for PDE with a variable coefficient. A parametrix can be used as an alternative for the PDE with variable coefficient. In contrast to the use of fundamental solution, the use of a parametrix does not bring down the BVPs to the Boundary Integral Equations (BIEs) that have lesser by one of dimensionality than the BVPs.

Nevertheless, by making use of the parametrix, the BVPs for PDEs with variable coefficients will be bring down to Boundary-Domain Integral Equations (BDIEs) that have the same dimensionality as the BVPs. For the discussions of BDIEs, please refer [3]-[6]. The preservation of the dimensionality of the BDIEs does not help the researchers in reduction of the cost of computation as the BDIEs need the discretization of the entire solution domain.

This disadvantage is in par with FEM. However, unlike BDIEs, FEM produces the system with sparse matrix. See e.g. [7]. This sparse matrix system of FEM is cheaper in computational cost as compared to the dense matrix system of BDIEs. See e.g. [7] and [2].

In year 2002, [8] used a localized parametrix as alternative to a parametrix for the BVP corresponds to PDE with variable coefficient. By utilizing the localized parametrix, the BVPs for PDEs with variable coefficient can be transformed to the localized Boundary-Domain Integral Equations (LBDIEs). The LBDIEs still need the discretization of the entire solution domain like in BDIEs but the prevailed system of equations is the sparse system of equations as obtained in FEM.

Even though the LBDIE enjoy the advantage of having sparse system of equations like in FEM, it however has a prominent drawback just like for BIE and BDIE. The drawback lies on the fact that some of the integrals in a BIE, BDIE and LBDIE are having singularities' expressions in part of their integrals' kernels. The singularities are of logarithmic singularity and   singularity.

There are several methods that have been proposed to handle both types of singularities. The standard procedure is by using suitable numerical integration formulae to integrate integrals with those singularities. The standard numerical integration formulae for integrals without singularity is Gauss quadrature. For the integrals involving logarithmic singularity, the use of Gauss Laguerre quadrature is more suitable. See e.g.[1]. However, this Gauss Laguerre quadrature formulae for integrals with singularity kernel may not give results as efficient as the Gauss quadrature for integrals without singularity kernel. Whereas, the Duffy transformation can be used for the singularities at the vertices for double integrals. Refer e.g. [1] and [9]. Other than Duffy transformation, [10] introduced a numerical method which is also based on transformation to reduce the order of singularity.

In 2012, [11] formulated a radial integration method transforming domain integrals into equivalent boundary integrals. The advantage of this technique is that the weak singularities as part of the integrations' kernels for the double integrals are transformed to the boundary integrals. Several results concerning radial integration methods can be found in [12]-[14].

In 2014, [15] derived a semi-analytic integration technique as an alternative to Gauss Laguerre quadrature formula for integrating the integrals with logarithmic singularity kernels.

In this work, we derive a method called semi quadratic integration method to integrate the integrals with   singularity as part of its integrals' kernels.

II. SEMI QUADRATIC INTEGRATION METHOD

A. Singularity

The second Green's theorem yields BIE for BVP of PDE with constant coefficient. One of the integrands of the boundary integrals is of the form $r^{-2} = |x-y|^{-2}$ which have singularity's influence in numerical computations. The singularity's influence is much obvious when the field point y is close to the source point x which is part of the integration element.

However, this singularity's influence can be handled easily by calculating the analytical solution for the condition whenever the field point y is close to the source point x which is part of the integration element. When the collocation point y is not part of the integration element, we can use standard numerical integration method, i.e. Gaussian quadrature.

Unfortunately, the BVP for PDE with variable coefficient $b(x)$ will not be bring down to BIEs like the constant coefficient case. Instead, it will be reduced to BDIEs or LBDIEs. Furthermore, the kernel of one of the BDIEs or LBDIEs is of the form $b(x)\dfrac{1}{r^2 b(y)}$.

The integration with the kernel $b(x)\dfrac{1}{r^2 b(y)}$ for which the field point y is close to the source point x which is part of the integration element can absolutely be calculated exactly. However, it is not practical to calculate the integration exactly for each and every time we have different values of variable coefficient $b(x)$.

B. Neumann Boundary Value Problem

We consider a linear elliptic PDE of order two with variable coefficient $b(x)$,

$$Lu(x) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} b(x) \frac{\partial}{\partial x_i} u(x) = f(x), \quad x \in \Omega,$$

where $u(x)$ for $x \in \partial\Omega \cup \Omega$ is the unknown, $f(x)$ and $b(x)$ are dictated functions, $\partial\Omega$ is the boundary and Ω is the domain.

Let's take the Neumann boundary condition as below.

$$Tu(x) = \sum_{i=1}^{2} b(x)\nu_i(x)\frac{\partial u(x)}{\partial x_i} = \bar{t}(x), \quad x \in \partial\Omega.$$

The notation $\nu(x) = (\nu_1(x), \nu_2(x))$ represents the outward normal to Ω.

Let's denote $P(x, y)$ as the parametrix which is given by

$$P(x,y) = \frac{\ln|x-y|}{2\pi b(y)}, \quad x, y \in \mathbb{R}^2.$$

Then, we have

$$T_x P(x,y) = \sum_{i=1}^{2} b(x) v_i(x) \frac{\partial P(x,y)}{\partial x_i} = \sum_{i=1}^{2} b(x) v_i(x) \frac{(x_i - y_i)}{2\pi b(y) r^2}.$$

By imposing a localized parametrix

$$P_\tau(x,y) = \tau(x,y) P(x,y),$$

we can transform a BVP for PDE with variable coefficient to a LBDIEs. Unlike a BDIE that produce a dense system of matrix equations, a LBDIEs will give rise to a system of sparse matrix equations.

Let's consider a constant cut-off function τ as given below,

$$\tau(x,y) = \begin{cases} 1, & x \in \omega_y, \\ 0, & x \notin \omega_y, \end{cases}$$

(1)

where ω_y is the localisation domain.

The localized parametrix $P_\tau(x,y)$ fulfills that

$$L_x P_\tau(x,y) = \delta(x-y) + R_\tau(x,y),$$

where $\delta(x-y)$ is the Dirac delta function and $R_\tau(x,y)$ is referred as

$$R_\tau(x,y) = R(x,y) + L_x((1-\tau)P).$$

Here $R(x,y)$ is the remainder which is defined as

$$R(x,y) = \frac{1}{2\pi b(y)} \sum_{i=1}^{2} \frac{x_i - y_i}{|y-x|} \frac{\partial b(x)}{\partial x_i}, \quad x, y \in \square .$$

The third Green identity localizes on $\omega(y) \cap \Omega$ and its boundary $\partial[\omega(y) \cap \Omega]$ is given below.

$$\begin{aligned} c(y)u(y) &- \int_{\bar{\omega}(y)\cap\partial\Omega} u(x) T_x P_\tau(x,y)\, d\Gamma(x) \\ &+ \int_{\bar{\omega}(y)\cap\partial\Omega} P_\tau(x,y) Tu(x)\, d\Gamma(x) \\ &- \int_{\Omega\cap\partial\omega(y)} u(x) T_x P_\tau(x,y)\, d\Gamma(x) \\ &+ \int_{\Omega\cap\partial\omega(y)} P_\tau(x,y) Tu(x)\, d\Gamma(x) \qquad (2) \\ &+ \int_{\omega(y)\cap\Omega} R_\tau(x,y) u(x)\, d\Omega(x) \\ &= \int_{\omega(y)\cap\Omega} P_\tau(x,y) f(x)\, d\Omega(x), \quad y \in \bar{\Omega}. \end{aligned}$$

The choice of cut-off function $\tau(x,y)$ in (1) implies

$$P_\tau(x,y) = \begin{cases} P(x,y), & x \in \omega_y, \\ 0, & x \in \omega_y, \end{cases}$$

and

$$R_\tau(x,y) = \begin{cases} R(x,y), & x \in \omega_y, \\ 0, & x \in \omega_y. \end{cases}$$

However, the Neumann BVP is not always solvable. If it does, the solvability of the Neumann BVP is unique up a constant [4].

III. RESULTS AND DISCUSSIONS

A. Discretization of the Neumann Localized Boundary-Domain Integral Equations

We substitute the Neumann boundary condition, $Tu(x) = \bar{t}(x),\ x \in \partial\Omega$, and add perturbation operator to Neumann LBDIEs (2) to get a unique solution for Neumann problem and obtain the perturbed BDIEs as follows.

$$\begin{aligned} & c(y)u(y) + \frac{1}{|\partial\Omega|} \int_{\partial\Omega} u(x)\, d\Gamma(x) \\ & - \int_{\partial\omega(y)} u(x) T_x P_\tau(x,y)\, d\Gamma(x) + \int_{\Omega\cap\partial\omega(y)} P_\tau(x,y) Tu(x)\, d\Gamma(x) \\ & + \int_{\omega(y)\cap\Omega} R_\tau(x,y) u(x)\, d\Omega(x) = -\int_{\bar{\omega}(y)\cap\partial\Omega} P_\tau(x,y) \bar{t}(x)\, d\Gamma(x) \\ & + \int_{\omega(y)\cap\Omega} P_\tau(x,y) f(x)\, d\Omega(x), \quad y \in \bar{\Omega}. \end{aligned}$$

We place field point x^i where $x^i \in \bar{\Omega}$ at J source points of the mesh and obtains the system of J equations.

$$c(x^i)u(x^i) + \sum_{x^j \in \bar{\omega}(x^i)} W_{ij} u(x^j) + \sum_{x^j \in \partial\Omega} \mathring{W}_{ij} u(x^j) = Q_i + D_i,\ x^i \in \bar{\Omega},$$

where W_{ij}, $\mathring{W}_{ij}$, Q_i and D_i are described as below.

$$\begin{aligned} W_{ij} &= -\int_{\partial\omega(x^i)} \phi_j(x) T_x P_\tau(x, x^i)\, d\Gamma(x) \\ &+ \int_{\Omega\cap\partial\omega(x^i)} P_\tau(x,x^i) \left[b(x) \frac{\partial \phi_j(x)}{\partial \nu(x)} \right] d\Gamma(x) \qquad (3) \\ &+ \int_{\omega(x^i)} \phi_j(x) R_\tau(x,x^i)\, d\Omega(x), \quad \text{if} \quad x^j \in \bar{\Omega}(x^i), \\ W_{ij} &= 0, \quad \text{if} \quad x^j \notin \bar{\Omega}(x^i), \end{aligned}$$

$$\mathring{W}_{ij} = \frac{1}{|\partial\Omega|} \int_{\partial\Omega} \phi_j(x)\, d\Gamma(x), \qquad (4)$$

$$Q_i = -\int_{\bar{\omega}(x^i)\cap\partial\Omega} P_\tau(x,x^i) \bar{t}(x)\, d\Gamma(x), \qquad (5)$$

$$D_i = \int_{\omega(x^i)} P_\tau(x,x^i) f(x)\, d\Omega(x). \qquad (6)$$

The global shape function ϕ_j are nonzero only on $\bar{\omega}(x^j)$, hence (3)-(6) yield

$$W_{ij} = -\sum_{\gamma_l \subset \bar{\varpi}(x^j) \cap \partial\omega(x^i)} \int_{\gamma_l} \phi_j(x) T_x P_\tau(x, x^i)\, \mathrm{d}\Gamma(x) + \sum_{\gamma_l \subset \bar{\varpi}(x^j) \cap \partial\omega(x^i) \cap \Omega} \int_{\gamma_l} P_\tau(x, x^i) \left[b(x) \frac{\partial \phi_j(x)}{\partial \nu(x)} \right] \mathrm{d}\Gamma(x) + \sum_{\Omega_m \subset \omega(x^i) \cap \omega(x^j)} \int_{\Omega_m} \phi_j(x) R_\tau(x, x^i)\, \mathrm{d}x, \quad (7)$$

$$\overset{\circ}{W}_{ij} = \frac{1}{|\partial\Omega|} \sum_{\partial\Omega_l \subset \bar{\varpi}(x^j)} \int_{\partial\Omega_l} \phi_j(x)\, \mathrm{d}\Gamma(x), \quad (8)$$

$$Q_i = -\sum_{\partial\Omega_l \subset \bar{\varpi}(x^i)} \int_{\partial\Omega_l} P_\tau(x, x^i) \bar{t}(x)\, \mathrm{d}\Gamma(x), \quad (9)$$

$$D_i = \sum_{\Omega_m \subset \omega(x^i)} \int_{\Omega_m} P_\tau(x, x^i) f(x)\, \mathrm{d}x. \quad (10)$$

Denoting

$$A_{n,i}^l = \int_{-1}^{1} \Psi_n(\eta) T_x P_\tau(x(\eta), x^i) J_{l1}(\eta)\, \mathrm{d}\eta, \quad (11)$$

$$C_{N,i}^l = \int_{-1}^{1} P_\tau(x(\eta), x^i) \mathcal{C}_N J_{l1}(\eta)\, \mathrm{d}\eta, \quad (12)$$

$$G_{N,i}^m = \int_{-1}^{1}\int_{-1}^{1} \Phi_N(\xi) R_\tau(x(\xi), x^i) J_{m2}(\xi) d\xi_1\, \mathrm{d}\xi_2, \quad (13)$$

$$F_i^l = \int_{-1}^{1} P_\tau(x(\eta), x^i) \bar{t}(x(\eta)) J_{l1}(\eta)\, \mathrm{d}\eta, \quad (14)$$

$$H_i^m = \int_{-1}^{1}\int_{-1}^{1} P_\tau(x(\xi), x^i) f(x(\xi)) J_{m1}(\xi)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2, \quad (15)$$

where

$$\mathcal{C}_N = \left[b(x(\eta)) \left(\sum_{p=1}^{2}\sum_{k=1}^{2} \frac{\partial \Phi_N(\xi)}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_p} \bigg|_{\xi=\xi(\eta)} \nu_p(x(\eta)) \right) \right],$$

and,

$\Psi_n(\eta)$ and $\Phi_n(\xi)$ are the local shape functions, respectively for 1D and 2D cases which are given by

$$\Psi_1(\eta) = \frac{1}{2}(1-\eta), \quad \Psi_2(\eta) = \frac{1}{2}(1+\eta), \quad -1 \le \eta \le 1,$$

$$\Phi_1(\xi) = (1-\xi_1)(1-\xi_2)/4, \quad \Phi_2(\xi) = (1+\xi_1)(1-\xi_2)/4,$$

$$\Phi_3(\xi) = (1+\xi_1)(1+\xi_2)/4, \quad \Phi_4(\xi) = (1-\xi_1)(1+\xi_2)/4,$$

$$-1 \le \xi_1, \xi_2 \le 1,$$

such that

$$x(\xi) = \sum_{N=1}^{4} \Phi_N(\xi) X^{mN}, \quad (16)$$

and

$$x(\eta) = \sum_{n=1}^{2} \Psi_n(\eta) X^{l\,n}. \quad (17)$$

Here X^{mN}, $N = 1,\ldots,4$ are the vertices for each domain element $e_m \subset \Omega$, $\bar{\Omega} = \bigcup_m^M \bar{e}_m$, $e_k \cap e_m = \varnothing$, $k \ne m$, , and $X^{l\,n}$, $n = 1,2$ are the end points for each boundary element $\partial\Omega_l \subset \partial\Omega$, $\partial\Omega = \cup_l^L \partial\Omega_l$. The notations J_{l1} and J_{m2} are the Jacobians of the transformation (16) and (17), respectively.

We can then write (7)-(10) as

$$W_{ij} = -\sum_{\gamma_l \subset \partial\omega(x^i) \cap \bar{\varpi}(x^j)} A_{n(j,l),i}^l + \sum_{\partial\omega_l(x^i) \subset \bar{\varpi}(x^j) \cap \Omega} C_{N(j,l),i}^l + \sum_{\Omega_m \subset \omega(x^i) \cap \omega(x^j)} G_{N(j,m),i}^m,$$

$$\overset{\circ}{W}_{ij} = \frac{1}{|\partial\Omega|} \sum_{\partial\Omega_l \subset \bar{\varpi}(x^j)} B_{n(j,l)}^l,$$

$$Q_i = -\sum_{\partial\omega_l \subset \bar{\varpi}(x^i)} F_i^l,$$

$$D_i = \sum_{\Omega_m \subset \bar{\varpi}(x^i)} H_i^m,$$

where $n(j,l)$ is the local number of the global source point x^j on the boundary element $\partial\Omega_l$ and $N(j,m)$ is the local number of the global source point x^j on the domain element Ω_m.

Note that $T_x P_\tau(x,y)$ in (11) has the singularity of $\frac{b(x(\eta))}{r^2 b(x^i)}$ when the field point x^i is close to the source point $x(\eta)$ which is part of the integration element. In this paper, we will derive a method to get rid of handle the singularity's influence of $\frac{b(x(\eta))}{r^2 b(x^i)}$. The method is named as quadratic semi- analytic method.

B. Derivation of the Quadratic Semi-Analytic Method

$A_{n,i}^l$ in (11) can be written as

$$A_{1i}^l = \int_{-1}^{1} \Psi_1(\eta) T_x P_\tau(x^i, x(\eta)) J_{l1}(\eta)\, \mathrm{d}\eta, \quad (18)$$

$$A_{2i}^l = \int_{-1}^{1} \Psi_2(\eta) T_x P_\tau(x^i, x(\eta)) J_{l1}(\eta)\, \mathrm{d}\eta, \quad (19)$$

where

$$T_x P_\tau(x^i, x(\eta)) = \frac{b(x(\eta))}{2\pi r^2 b(x^i)} \left((x_1 - x_1^i)\nu_1(x) + (x_2 - x_2^i)\nu_2(x) \right).$$

Equations (17) leads us to the linear interpolation of $b(x(\eta))$ as given below.

$$b(x(\eta)) \approx \Psi_1(\eta) b(x(-1)) + \Psi_2(\eta) b(x(1)). \tag{20}$$

Multiplying $\Psi_1(\eta)$ and $\Psi_2(\eta)$, respectively to (20) give

$$\Psi_1(\eta) b(x(\eta)) \approx \left(\Psi_1(\eta)\right)^2 b(x(-1)) + \Psi_1(\eta)\Psi_2(\eta) b(x(1)),$$

$$\Psi_2(\eta) b(x(\eta)) \approx \Psi_1(\eta)\Psi_2(\eta) b(x(-1)) + \left(\Psi_2(\eta)\right)^2 b(x(1)).$$

The expressions A_{1i}^l and A_{2i}^l in (18) and (19) can be set up as given below.

$$A_{1i}^l = G_{B1}^l + G_{A1}^l, \quad A_{2i}^l = G_{B2}^l + G_{A2}^l,$$

where

$$g_{b1} = \left(\Psi_1(\eta)\right)^2 T_x P_\tau(x^i, x(-1)) + \Psi_1(\eta)\Psi_2(\eta) T_x P_\tau(x^i, x(1)),$$

$$g_{b2} = \Psi_1(\eta)\Psi_2(\eta) T_x P_\tau(x^i, x(-1)) + \left(\Psi_2(\eta)\right)^2 T_x P_\tau(x^i, x(1)),$$

$$T_x P_\tau(x^i, x(-1)) = \left(\frac{b(x(-1))}{2\pi r^2 b(x^i)}\right) \times \left((x_1(-1) - x_1^i)\nu_1(x) + (x_2(-1) - x_2^i)\nu_2(x)\right),$$

$$T_x P_\tau(x^i, x(1)) = \left(\frac{b(x(1))}{2\pi r^2 b(x^i)}\right) \times \left((x_1(1) - x_1^i)\nu_1(x) + (x_2(1) - x_2^i)\nu_2(x)\right),$$

$$G_{B1}^l = \int_{-1}^{1} \left(\Psi_1(\eta) T_x P_\tau(x^i, x(\eta)) - g_{b1}\right) J_{l1}(\eta)\, d\eta, \tag{21}$$

$$G_{B2}^l = \int_{-1}^{1} \left(\Psi_2(\eta) T_x P_\tau(x^i, x(\eta)) - g_{b2}\right) J_{l1}(\eta)\, d\eta. \tag{22}$$

$$\begin{aligned} G_{A1}^l &= \int_{-1}^{1} g_{b1} J_{l1}(\eta)\, d\eta \\ &= \int_{-1}^{1} \left(\Psi_1(\eta)\right)^2 T_x P_\tau(x^i, x(-1)) J_{l1}(\eta)\, d\eta \\ &\quad + \int_{-1}^{1} \Psi_1(\eta)\Psi_2(\eta) T_x P_\tau(x^i, x(1)) J_{l1}(\eta)\, d\eta \\ &= \left(\frac{b(x(-1))\left((x_1(-1) - x_1^i)\nu_1(x) + (x_2(-1) - x_2^i)\nu_2(x)\right)}{8\pi b(x^i)}\right) \times \int_{-1}^{1} (1-\eta)^2 \left(\frac{1}{r^2}\right) J_{l1}(\eta)\, d\eta \\ &\quad + \left(\frac{b(x(1))\left((x_1(1) - x_1^i)\nu_1(x) + (x_2(1) - x_2^i)\nu_2(x)\right)}{8\pi b(x^i)}\right) \times \int_{-1}^{1} (1-\eta)(1+\eta) \left(\frac{1}{r^2}\right) J_{l1}(\eta)\, d\eta, \end{aligned} \tag{23}$$

$$\begin{aligned} G_{A2}^l &= \int_{-1}^{1} g_{b2} J_{l1}(\eta)\, d\eta \\ &= \int_{-1}^{1} \Psi_1(\eta)\Psi_2(\eta) T_x P_\tau(x^i, x(-1)) J_{l1}(\eta)\, d\eta \\ &\quad + \int_{-1}^{1} \left(\Psi_2(\eta)\right)^2 T_x P_\tau(x^i, x(1)) J_{l1}(\eta)\, d\eta \\ &= \left(\frac{b(x(-1))\left((x_1(-1) - x_1^i)\nu_1(x) + (x_2(-1) - x_2^i)\nu_2(x)\right)}{8\pi b(x^i)}\right) \times \int_{-1}^{1} (1-\eta)(1+\eta) \left(\frac{1}{r^2}\right) J_{l1}(\eta)\, d\eta \\ &\quad + \left(\frac{b(x(1))\left((x_1(1) - x_1^i)\nu_1(x) + (x_2(1) - x_2^i)\nu_2(x)\right)}{8\pi b(x^i)}\right) \times \int_{-1}^{1} (1+\eta)^2 \left(\frac{1}{r^2}\right) J_{l1}(\eta)\, d\eta. \end{aligned} \tag{24}$$

The integrals G_{B1} and G_{B2} in (21) and (22) are computed by numerical quadrature i.e. the standard Gaussian quadrature. Whereas, the integrals G_{A1} and G_{A2} in equations (23) and (24), respectively, will be computed analytically. The analytical expressions of G_{A1} and G_{A2} are as follows.

$$G_{A1} = \frac{b(x(-1))\left((x_1(-1) - x_1^i)\nu_1(x) + (x_2(-1) - x_2^i)\nu_2(x)\right)}{8\pi b(x^i)} g_{A1} + \frac{b(x(1))\left((x_1(1) - x_1^i)\nu_1(x) + (x_2(1) - x_2^i)\nu_2(x)\right)}{8\pi b(x^i)} g_{A2},$$

$$G_{A2} = \frac{b(x(-1))\left((x_1(-1)-x_1^i)\nu_1(x)+(x_2(-1)-x_2^i)\nu_2(x)\right)}{8\pi b(x^i)} g_{A2} + \frac{b(x(1))\left((x_1(1)-x_1^i)\nu_1(x)+(x_2(1)-x_2^i)\nu_2(x)\right)}{8\pi b(x^i)} g_{A3},$$

where

$$g_{A1} = \int_{-1}^{1} \frac{(1-\eta)^2}{r^2}\frac{ds}{d\eta}\,\mathrm{d}\eta, \qquad (25)$$

$$g_{A2} = \int_{-1}^{1} \frac{(1+\eta)(1-\eta)}{r^2}\frac{ds}{d\eta}\,\mathrm{d}\eta, \qquad (26)$$

$$g_{A3} = \int_{-1}^{1} \frac{(1+\eta)^2}{r^2}\frac{ds}{d\eta}\,\mathrm{d}\eta.$$

(27)

Analytically the integrals in (25)-(27) can be written as follows.

$$g_{A1} = \left(\frac{8J_{l1}(\eta)}{V_2^4}\right)\times \left(V_2^2 - \frac{(V_1^2V_2^2 - V_2^4 + 2V_2^2V_3 - 2V_3^2)g_1}{\sqrt{V_1^2V_2^2 - V_3^2}} + (-V_2^2 + V_3)\,\mathrm{g}_2\right),$$

$$g_{A2} = \left(\frac{8J_{l1}(\eta)\left(V_1^2V_2^2 + (V_2^2 - 2V_3)V_3\right)g_1}{V_2^4\sqrt{V_1^2V_2^2 - V_3^2}}\right) + \left(\frac{4J_{l1}(\eta)\sqrt{V_1^2V_2^2 - V_3^2}\left(-2V_2^2 + (V_2^2 - 2V_3)g_2\right)}{V_2^4\sqrt{V_1^2V_2^2 - V_3^2}}\right),$$

$$g_{A3} = \frac{8J_{l1}(\eta)\left((-V_1^2)V_2^2 + 2V_3^2\right)g_1}{V_2^4\sqrt{V_1^2V_2^2 - V_3^2}} + \frac{8J_{l1}(\eta)\sqrt{V_1^2V_2^2 - V_3^2}\left(V_2^2 + V_3 g_2\right)}{V_2^4\sqrt{V_1^2V_2^2 - V_3^2}},$$

where

$$J_{l1}(\eta) = \frac{ds}{d\eta} = \frac{V_2}{2},$$

$$g_1 = ArcTan\left[\frac{\sqrt{V_1^2V_2^2 - V_3^2}}{(V_1^2 - V_3)}\right],$$

$$g_2 = Log\left[\frac{(V_1^2 + V_2^2 - 2V_3)}{V_1^2}\right],$$

and V_1, V_2 and V_3 are defined below.

$$V_1 = | x^i - x(s_{l1}) |,$$
$$V_2 = | x(s_{l2}) - x(s_{l1}) |,$$
$$V_3 = (x^i - x(s_{l1}))\cdot(x(s_{l2}) - x(s_{l1})).$$

IV. CONCLUSION

The problem of having the influence of the $r^{-2} = | x-y |^{-2}$ singularity of $T_x P_\tau(x,y)$ in one of the LBDIEs can be handled by representing the boundary integral as two separate integrals. Respectively, integral with the variable coefficient $b(x)$ and without the variable coefficient $b(x)$. The one without the variable coefficient $b(x)$ can be calculated exactly since the integrand is not varies even for different values of variable coefficient $b(x)$. The one with variable coefficient $b(x)$ is to be calculated numerically.

It is clear that this idea can overcome the influence of the singularity $r^{-2} = | x-y |^{-2}$ based on the fact that the exact solution obtained for integral with the variable coefficient $b(x)$ moves faster than its numerical solution. For the future research, we aim to do some numerical experiment to validate our claim.

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